Approximate Solution for the System of Non-linear Volterra Integral Equations of the Second Kind by Five Block-by-block Method

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Abstract: In this paper, the block method modified and applied to find a numerical solutions for the system of non-linear Volterra integral equations of the second kind (NSVIEK2). Since it is an efficient numerical method for solving this system, which avoids the need for special starting values, the method has also the advantages of simplicity of application and gives high accuracy which is clarified by numerical examples.

Keywords: five block-by-block, system of non-linear Volterra integral equations

Introduction:

The block- by -block method is used to obtain the numerical solution of this system .It is essentially an extrapolation procedure which has advantage of being self-starting and produces a block of values at a time (Delves and Mohamed, 1985), (Delves and Walsh, 1974).

Different method are used to solved integral equations which are investigated from many physical application such as the mixed problem in the theory of elasticity.

(Linz, 1969) describes two block-by-block method and used this method to solve Volterra integral equations of the second kind. The method given in this paper is based upon the modified block-by-block method of Linz.

(Saeed and Ahmed, 2008) describes two and three blocks method and used this methods to solve a system of non-linear Volterra integral equations of the second kind.

In this paper, the approaches of block is modified to five blocks and applied for finding a numerical solution of NSVIEK2's, in which a block of five values are produced at each stage, and these values are obtained using the five-point quadrature formula.

Block-by-block method:

The idea behind the block-by-block method is to divide the interval [a, b] in to mesh-point $x_j = jh$, $j = 0,1,\ldots,n$ and then we try to evaluate the value of the unknown function $u_i(x)$ at these mesh-points.

For solving system of non-linear Volterra integral equations:

$$U(x) = F(x) + \int_0^x K(x,t,U(t))dt\,, \qquad (1)$$

Rewrite equation (1) as follows:

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$$u_i(x_k) = f_i(x_k) + \int_a^{x_{pl}} k_i(x_k, t, U(t))dt + \int_{x_{pl}}^{x_k} k_i(x_k, t, U(t))dt\,, \qquad (2)$$

If the values u_{i0}, u_{i1}, ..., $u_{i,pl}$ are known, then the first integral can be approximated by standard quadrature methods. The second integral is estimated by a quadrature rule using values of the integrand at $t = x_{pl}, x_{pl+1}, \ldots, x_{p(l+1)}$. Since the values of u_i at these points are unknown, we have a system of *mp* non-linear simultaneous equations

$$u_{ik} = f_i(x_k) + h\sum_{j=0}^{lp} w_{kj} k_i(x_k, t_j, u_{1j}, \ldots, u_{mj})$$

$$+ h\sum_{j=0}^{p} w'_{kj} k_i(x_k, t_{lp+j}, u_{1,lp+j}, \ldots, u_{m,lp+j})\,, \qquad (3)$$

for $k = lp+1, lp+2, \ldots, (l+1)p$, where $w_{kj}, w_{kj}^{'}$ depend on the quadrature rule used. For sufficiently small *h* the system we obtain from equation (3) has a unique solution which can be determined by iteration method. Thus, a 'block' of *p* values of u_i is obtained simultaneously. We derive a new types of block method which is five block method.

Application of the Modified Block-By-Block Method on the NSVIEK2:

For this method we take $p = 5$ the integration over $[a, x_{5lr}]$, *r*=0, 1, 2, 3, 4, 5, can be accomplished by using a quadratic interpolation of the integrand at the point $t_{5l}, t_{5l+1}, t_{5l+2}, t_{5l+3}, t_{5l+4}, t_{5l+5}$. Then equation (1) becomes:

$$u_{i,5l+1} = f_i(x_{5l+1}) + \int_a^{(5l+1)h} k_i(x_{5l+1}, t, U(t))dt\,, \qquad (4)$$

$$u_{i,5l+2} = f_i(x_{5l+2}) + \int_a^{(5l+2)h} k_i(x_{5l+2}, t, U(t))dt\,, \qquad (5)$$

$$u_{i,5l+3} = f_i(x_{5l+3}) + \int_a^{(5l+3)h} k_i(x_{5l+3}, t, U(t))dt\,, \qquad (6)$$

$$u_{i,5l+4} = f_i(x_{5l+4}) + \int_a^{(5l+4)h} k_i(x_{5l+4}, t, U(t))dt\,, \qquad (7)$$

$$u_{i,5l+5} = f_i(x_{5l+5}) + \int_a^{(5l+5)h} k_i(x_{5l+5}, t, U(t))dt\,. \qquad (8)$$

where, $i = 1, 2, \ldots, m$, $l = 0, 1, \ldots$..

From equation (2), equation (4) - (8) can be written as:

$$u_{i,5l+1} = f_i(x_{5l+1}) + \int_a^{5lh} k_i(x_{5l+1}, t, U(t))dt + \int_{5lh}^{(5l+1)h} k_i(x_{5l+1}, t, U(t))dt \qquad (9)$$

$$u_{i,5l+2} = f_i(x_{5l+2}) + \int_a^{5lh} k_i(x_{5l+2}, t, U(t))dt + \int_{5lh}^{(5l+2)h} k_i(x_{5l+2}, t, U(t))dt \qquad (10)$$

$$u_{i,5l+3} = f_i(x_{5l+3}) + \int_a^{5lh} k_i(x_{5l+3}, t, U(t))dt + \int_{5lh}^{(5l+3)h} k_i(x_{5l+3}, t, U(t))dt \qquad (11)$$

$$u_{i,5l+4} = f_i(x_{5l+4}) + \int_a^{5lh} k_i(x_{5l+4}, t, U(t))dt + \int_{5lh}^{(5l+4)h} k_i(x_{5l+4}, t, U(t))dt \qquad (12)$$

$$u_{i,5l+5} = f_i(x_{5l+5}) + \int_a^{5lh} k_i(x_{5l+4}, t, U(t))dt + \int_{5lh}^{(5l+5)h} k_i(x_{5l+5}, t, U(t))dt \ . \qquad (13)$$

Setting s=1/2, n=4 and using equation (1) and (8) in (Saeed and Ahmed, 2008) the last term in equations (9), (10), (11) and (12) becomes:

$$\int_{x_{5l+q}}^{x_{5l+(q+1)}} f(x)dx = \frac{h}{6}\Big[k_i(x_{5l+(q+1)}, t_{5l+q}, u_{1,5l+q}, \ldots, u_{m,5l+q}) + 4k_i\Big(x_{5l+(q+1)}, t_{5l+(q+0.5)},$$
$$\frac{1}{256}(63u_{1,5l} + 315u_{1,5l+1} - 210u_{1,5l+2} + 126u_{1,5l+3} - 45u_{i,5l+4} + 7u_{i,5l+5}), \ldots,$$
$$\frac{1}{256}(63u_{m,5l} + 315u_{m,5l+1} - 210u_{m,5l+2} + 126u_{m,5l+3} - 45u_{m,5l+4} + 7u_{m,5l+5})\Big)$$
$$+ k_i(x_{5l+(q+1)}, t_{5l+(q+1)}, u_{1,5l+(q+1)}, \ldots, u_{m,5l+(q+1)})\Big],$$

where q=0, 1, 2, 3 (14)

From equation (23), we get a new formula of five blocks as follows:

If l is even

$$u_{i,5l+1} = f_i(x_{5l+1}) + \frac{h}{3}\sum_{j=0}^{5l} w_j k_i(x_{5l+1}, t_j, u_{1j}, \ldots, u_{mj}) + \frac{h}{6}\Big[k_i(x_{5l+2}, t_{5l+1}, u_{1,5l+1}, \ldots, u_{m,5l+1})$$
$$+ 4k_i\Big(x_{5l+1}, t_{5l+0.5}, \frac{1}{256}\big(63u_{1,5l} + 315u_{1,5l+1} - 210u_{1,5l+2} + 126u_{1,5l+3} - 45u_{1,5l+4}$$
$$+ 7u_{1,5l+5}\big), \ldots, \frac{1}{256}\big(63u_{m,5l} + 315u_{m,5l+1} - 210u_{m,5l+2} + 126u_{m5l+3} - 45u_{m,5l+4} + 7u_{m,5l+5}\big)\Big)$$
$$+ k_i\big(x_{5l+1}, t_{5l+1}, u_{1,5l+1}, \ldots, u_{m,5l+1}\big)\Big] \qquad (15)$$

$$u_{i,5l+2} = f_i(x_{5l+2}) + \frac{h}{3}\sum_{j=0}^{5l} w_j k_i(x_{5l+2}, t_j, u_{1j}, \ldots, u_{mj}) + \frac{h}{2}\sum_{j=5l}^{5l+1} k_i(x_{5l+2}, t_j, u_{1j}, \ldots, u_{mj})$$
$$+ \frac{h}{6}\Big[k_i\big(x_{5l+2}, t_{5l+1}, u_{1,5l+1}, \ldots, u_{m,5l+1}\big) + 4k_i\Big(x_{5l+2}, t_{5l+1.5}, \frac{1}{256}\big(63u_{1,5l} + 315u_{1,5l+1}$$
$$_i - 210u_{1,5l+2} + 126u_{1,5l+3} - 45u_{1,5l+4} + 7u_{1,5l+5}\big), \ldots, \frac{1}{256}\big(63u_{m,5l} + 315u_{m,5l+1}$$
$$- 210u_{m,5l+2} + 126u_{m,5l+3} - 45u_{m,5l+4} + 7u_{m,5l+5}\big)\Big) + k_i(x_{5l+2}, t_{5l+2}, u_{1,5l+2}, \ldots, u_{m,5l+2})\Big]$$

(16)

$$u_{i,5l+3} = f_i(x_{5l+3}) + \frac{h}{3}\sum_{j=0}^{5l+2} \overline{w}_j k_i(x_{5l+3}, t_j, u_{1j}, \ldots, u_{mj}) + \frac{h}{6}\Big[k_i(x_{5l+3}, t_{5l+2}, u_{1,5l+2}, \ldots, u_{m,5l+2})$$

$$+4k_i\left(x_{5l+3}, t_{5l+2.5}, \frac{1}{256}\left(63u_{1,5l}+315u_{1,5l+1}-210u_{1,5l+2}+126u_{1,5l+3}-45u_{1,5l+4}\right.\right.$$

$$\left.\left.+7u_{1,5l+5}\right),\ldots,\frac{1}{256}\left(63u_{m,5l}+315u_{m,5l+1}-210u_{m,5l+2}+126u_{m,5l+3}-45u_{m,5l+4}+7u_{m,5l+5}\right)\right)$$

$$\left.+k_i\left(x_{5l+3}, t_{5l+3}, u_{1,5l+3},\ldots,u_{m,5l+3}\right)\right]. \tag{17}$$

$$u_{i,5l+4} = f_i(x_{5l+4}) + \frac{h}{3}\sum_{j=0}^{5l+2}\overline{w}_j k_i(x_{5l+4}, t_j, u_{1j},\ldots,u_{mj}) + \frac{h}{2}\sum_{j=5l+2}^{5l+3} k_i(x_{5l+4}, t_j, u_{1j},\ldots,u_{mj})$$

$$+\frac{h}{6}\left[k_i\left(x_{5l+4}, t_{5l+3}, u_{1,5l+3},\ldots,u_{m,5l+3}\right)+4k_i\left(x_{5l+4}, t_{5l+3.5}, \frac{1}{256}\left(63u_{1,5l}+315u_{1,5l+1}\right.\right.\right.$$

$$_i-210u_{1,5l+2}+126u_{1,5l+3}-45u_{1,5l+4}+7u_{1,5l+5}\Big),\ldots,\frac{1}{256}\Big(63u_{m,5l}+315u_{m,5l+1}$$

$$-210u_{m,5l+2}+126u_{m,5l+3}-45u_{m,5l+4}+7u_{m,5l+5}\Big)\Big)$$

$$\left.+k_i(x_{5l+4}, t_{5l+4}, u_{1,5l+4},\ldots,u_{m,5l+4})\right]. \tag{18}$$

$$u_{i,5l+5} = f_i(x_{5l+5}) + \frac{h}{3}\sum_{j=0}^{5l+4} Z_j k_i(x_{5l+5}, t_j, u_{1j},\ldots,u_{mj}) + \frac{h}{2}\sum_{j=5l+4}^{5l+5} k_i(x_{5l+5}, t_j, u_{1j},\ldots,u_{mj})\,. \tag{19}$$

If l is odd

$$u_{i,5l+1} = f_i(x_{5l+1}) + \frac{h}{3}\sum_{j=0}^{5l} w_j k_i(x_{5l+1}, t_j, u_{1j},\ldots,u_{mj}) + \frac{h}{6}\left[k_i(x_{5l+2}, t_{5l+1}, u_{1,5l+1},\ldots,u_{m,5l+1})\right.$$

$$+4k_i\left(x_{5l+1}, t_{5l+0.5}, \frac{1}{256}\left(63u_{1,5l}+315u_{1,5l+1}-210u_{1,5l+2}+126u_{1,5l+3}-45u_{1,5l+4}\right.\right.$$

$$\left.\left.+7u_{1,5l+5}\right),\ldots,\frac{1}{256}\left(63u_{m,5l}+315u_{m,5l+1}-210u_{m,5l+2}+126u_{m,5l+3}-45u_{m,5l+4}+7u_{m,5l+5}\right)\right)$$

$$\left.+k_i\left(x_{5l+1}, t_{5l+1}, u_{1,5l+1},\ldots,u_{m,5l+1}\right)\right]. \tag{20}$$

$$u_{i,5l+2} = f_i(x_{5l+2}) + \frac{h}{3}\sum_{j=0}^{5l+1}\overline{Z}_j k_i(x_{5l+2}, t_j, u_{1j},\ldots,u_{mj}) + \frac{h}{6}\left[k_i\left(x_{5l+2}, t_{5l+1}, u_{1,5l+1},\ldots,u_{m,5l+1}\right)\right.$$

$$+4k_i\left(x_{5l+2}, t_{5l+1.5}, \frac{1}{256}\left(63u_{1,5l}+315u_{1,5l+1}-210u_{1,5l+2}+126u_{1,5l+3}-45u_{1,5l+4}+7u_{1,5l+5}\right),\right.$$

$$\left.\ldots,\frac{1}{256}\left(63u_{m,5l}+315u_{m,5l+1}-210u_{m,5l+2}+126u_{m,5l+3}-45u_{m,5l+4}+7u_{m,5l+5}\right)\right)$$

$$\left.+k_i(x_{5l+2}, t_{5l+2}, u_{1,5l+2},\ldots,u_{m,5l+2})\right]. \tag{21}$$

$$u_{i,5l+3} = f_i(x_{5l+3}) + \frac{h}{2}\sum_{j=0}^{5l+2}\overline{w}_j k_i(x_{5l+3}, t_j, u_{1j},\ldots,u_{mj}) + \frac{h}{6}\left[k_i(x_{5l+3}, t_{5l+2}, u_{1,5l+2},\ldots,u_{m,5l+2})\right.$$

$$+4k_i\left(x_{5l+3},t_{5l+2.5},\frac{1}{256}\left(63u_{1,5l}+315u_{1,5l+1}-210u_{1,5l+2}+126u_{1,5l+3}-45u_{1,5l+4}\right.\right.$$

$$\left.\left.+7u_{1,5l+5}\right),\ldots,\frac{1}{256}\left(63u_{m,5l}+315u_{m,5l+1}-210u_{m,5l+2}+126u_{m,5l+3}-45u_{m,5l+4}+7u_{m,5l+5}\right)\right)$$

$$+k_i\left(x_{5l+3},t_{5l+3},u_{1,5l+3},\ldots,u_{m,5l+3}\right)\Big]. \tag{22}$$

$$u_{i,5l+4}=f_i(x_{5l+4})+\frac{h}{3}\sum_{j=0}^{5l+3}M_jk_i(x_{5l+4},t_j,u_{1j},\ldots,u_{mj})+$$

$$\frac{h}{6}\Big[k_i\left(x_{5l+4},t_{5l+3},u_{1,5l+3},\ldots,u_{m,5l+3}\right)+4k_i\left(x_{5l+4},t_{5l+3.5},\frac{1}{256}\left(63u_{1,5l}+315u_{1,5l+1}\right.\right.$$

$$_i-210u_{1,5l+2}+126u_{1,5l+3}-45u_{1,5l+4}+7u_{1,5l+5}\Big),\ldots,\frac{1}{256}\left(63u_{m,5l}+315u_{m,5l+1}\right.$$

$$\left.\left.\left.-210u_{m,5l+2}+126u_{m,5l+3}-45u_{m,5l+4}+7u_{m,5l+5}\right)\right)\right]$$

$$+k_i(x_{5l+4},t_{5l+4},u_{1,5l+4},\ldots,u_{m,5l+4})\Big]. \tag{23}$$

$$u_{i,5l+5}=f_i(x_{5l+5})+\frac{h}{3}\sum_{j=0}^{5l+5}\overline{M}_jk_i(x_{5l+5},t_j,u_{1j},\ldots,u_{mj}). \tag{24}$$

where $w_0=w_{5l}=1$, $w_j=3-(-1)^j$, $j=1,2,\ldots,5l-1$

$\overline{Z}_0=\overline{Z}_{5l+1}=1$, $\overline{Z}_j=3-(-1)^j$, $j=1,2,\ldots,5l$

$\overline{w}_0=\overline{w}_{5l+2}=1$, $\overline{w}_j=2$, $j=1,2,\ldots,5l+1$

$M_0=M_{5l+3}=1$, $M_j=3-(-1)^j$, $j=1,2,\ldots,5l+2$

$Z_0=Z_{5l+4}=1$, $Z_j=3-(-1)^j$, $j=1,2,\ldots,5l+3$

$\overline{M}_0=\overline{M}_{5l+5}=1$, $\overline{M}_j=3-(-1)^j$, $j=1,2,\ldots,5l+4$,

$i=1,2,\ldots,m$, $l=0,1,\ldots$

Therefore, at each step we construct $5m$ simultaneously non-linear equations from the equation (15)-(24) which can be solved for the unknown's $u_{i,5l+1}$, $u_{i,5l+2}$, $u_{i,5l+3}$, $u_{i,5l+4}$ and $u_{i,5l+5}$, $i=1,2,\ldots,m$ by using modified Newton-Raphson method.

Illustrative Examples:

We demonstrating the method by three examples, and depending on the least square errors to comparison among the solutions obtained by this method against the exact solution.

Example 1: (Babolian, 2000)

Solve a system of non-linear VIEK2's:

$$u_1(x)=x-x^2+\int_0^x(u_1(t)+u_2(t))dt$$

$$u_2(x) = x - \frac{1}{2}x^2 - \frac{1}{3}x^3 + \int_0^x (u_1^2(t) + u_2(t))dt$$

The exact solution of this system is:

$u_1(x) = x$ and $u_2(x) = x$.

Table (1) comparison between the exact solution and the numerical solution $u_1(x)$ and $u_2(x)$ of Example 1 taking h=0.1.

X	Exact solution	$u_1(x)$	$u_2(x)$
		MBLM5	MBLM5
0	0	0	0
0.1	0.1	0.100400786	0.100223444
0.2	0.2	0.186747619	0.192189706
0.3	0.3	0.271981958	0.281706785
0.4	0.4	0.354335758	0.367951400
0.5	0.5	0.484654127	0.487865446
0.6	0.6	0.541568504	0.552669537
0.7	0.7	0.650794392	0.651684964
0.8	0.8	0.617353475	0.638621788
0.9	0.9	0.789899616	0.776894486
1	1	0.902470729	0.884351544
L.S.E.		6.4110×10^{-02}	6.0717×10^{-02}

Table (2) shows the least square errors for $u_1(x)$ and $u_2(x)$ with different values of h for Example 1.

Numerical solution of	least square errors		
	h=0.1	h=0.05	H=0.025
$u_1(x)$	6.4110×10^{-02}	2.2870×10^{-03}	1.2124×10^{-04}
$u_2(x)$	6.0717×10^{-02}	1.0959×10^{-03}	4.1972×10^{-05}

Example 2: (Waswas, 2005)

Solve a system of non-linear VIEK2's:

$$u_1(x) = \sec(x) - x + \int_0^x \left((u_1(t))^2 - (u_2(t))^2\right)dt$$

$$u_2(x) = 3\tan(x) - x - \int_0^x \left((u_1(t))^2 + (u_2(t))^2\right)dt$$

The exact solution of this system is:

$u_1(x) = \sec(x)$ and $u_2(x) = \tan(x)$

Table (3) comparison between the exact solution and the numerical solution $u_1(x)$ and $u_2(x)$ of Example 2 taking h=0.1.

x	$u_1(x)$		$u_2(x)$	
	Exact solution	MBLM5	Exact solution	MBLM5
0	1	1.000000000	0	0
0.1	1.005020918	1.023717242	0.100334672	0.081852519
0.2	1.020338845	1.041962318	0.202710036	0.184249876
0.3	1.046751602	1.073984121	0.309336249	0.292510872
0.4	1.085704428	1.154587439	0.422793219	0.375706644
0.5	1.139493927	1.179585514	0.546302489	0.519203066
0.6	1.211628315	1.200693001	0.684136808	0.698986226
0.7	1.307459259	1.340257166	0.842288380	0.829283277
0.8	1.435324199	1.383529392	1.029638557	1.266835501
0.9	1.608725810	1.617070186	1.260158218	1.312670413
1	1.850815718	1.857619760	1.557407725	1.477949519
L.S.E.		1.1904×10^{-02}		6.9640×10^{-02}

Table (4) shows the least square errors for $u_1(x)$ and $u_2(x)$ with different values of h for Example 2.

Numerical solution of	least square errors		
	h=0.1	h=0.05	H=0.025
$u_1(x)$	1.1904×10^{-02}	3.3693×10^{-03}	8.9688×10^{-04}
$u_2(x)$	6.9640×10^{-02}	4.7832×10^{-03}	9.7342×10^{-04}

Example 3: (Jumaa, 2005)

Solve a system of non-linear VIEK2's:

$$u_1(x) = \frac{1}{4} - \frac{1}{4}e^{2x} + \int_0^x (x-t)u_2^2(t)dt$$

$$u_2(x) = -xe^x + 2e^x - 1 + \int_0^x te^{-2u_1(t)}dt$$

The exact solution of this system is:

$$u_1(x) = -\frac{1}{2}x \quad \text{and} \quad u_2(x) = e^x$$

Table (5) comparison between the exact solution and the numerical solution $u_1(x)$ and $u_2(x)$ of Example 3 taking h=0.1.

X	$u_1(x)$		$u_2(x)$	
	Exact solution	MBLM5	Exact solution	MBLM5
0	0	0	1	1.000000000
0.1	-0.05000000	-0.049996538	1.105170918	1.107403520
0.2	-0.10000000	-0.101093072	1.221402758	1.227183482
0.3	-0.15000000	-0.151609850	1.349858808	1.357191484
0.4	-0.20000000	-0.196264598	1.491824698	1.499169859
0.5	-0.25000000	-0.325424028	1.648721271	1.655593426
0.6	-0.30000000	-0.320934408	1.822118800	1.915247797
0.7	-0.35000000	-0.348572145	2.013752707	2.154858565
0.8	-0.40000000	-0.411246275	2.225540928	2.086177826
0.9	-0.45000000	-0.434948422	2.459603111	2.615156470
1	-0.50000000	-0.768179020	2.718281828	2.796935599
L.S.E.		7.8419×10^{-02}		7.8582×10^{-02}

Table (6) shows the least square errors for $u_1(x)$ and $u_2(x)$ with different values of h for Example 3.

Numerical solution of	least square errors		
	h=0.1	h=0.05	H=0.025
$u_1(x)$	7.8419×10^{-02}	1.7864×10^{-03}	6.7330×10^{-05}
$u_2(x)$	7.8582×10^{-02}	9.7120×10^{-04}	3.7381×10^{-05}

Conclusions:

We used the MBLM2, MBLM3 (Saeed, R. K. and Ahmed, C. S., (2008)) and MBLM5 to solve this system is very efficient we concluded by the numerical results which obtaining from the illustrative examples, the method produce a good accuracy if the function $f_i(x)$, i=1, 2, ..., m are polynomial, also for sufficiently small h since by reducing step size length the least square error will be reduced also the small value of p since by using a few equations the least square error will be reduced. The comparison between them showing in this tables:

Table (7) shows the least square errors for $u_1(x)$ and $u_2(x)$ with different values of h for Example 1.

Numerical solution of	methods	least square errors		
		h=0.1	h=0.05	h=0.025
$u_1(x)$	MBLM2	0	0	0
	MBLM3	1.1293×10^{-02}	3.7858×10^{-04}	1.9642×10^{-05}
	MBLM5	6.4110×10^{-02}	2.2870×10^{-03}	1.2124×10^{-04}
$u_2(x)$	MBLM2	0	0	0
	MBLM3	1.1167×10^{-02}	1.8703×10^{-04}	6.8744×10^{-06}
	MBLM5	6.0717×10^{-02}	1.0959×10^{-03}	4.1972×10^{-05}

Table (8) shows the least square errors for $u_1(x)$ and $u_2(x)$ with different values of h for Example 2.

Numerical solution of	methods	least square errors		
		h=0.1	h=0.05	h=0.025
$U_1(x)$	MBLM2	0	0	0
	MBLM3	1.1293×10^{-02}	3.7858×10^{-04}	1.9642×10^{-05}
	MBLM5	1.1904×10^{-02}	3.3693×10^{-03}	8.9688×10^{-04}
$U_2(x)$	MBLM2	0	0	0
	MBLM3	1.1167×10^{-02}	1.8703×10^{-04}	6.8744×10^{-06}
	MBLM5	6.9640×10^{-02}	4.7832×10^{-03}	9.7342×10^{-04}

Table (9) shows the least square errors for $u_1(x)$ and $u_2(x)$ with different values of h for Example 3.

Numerical solution of	methods	least square errors		
		h=0.1	h=0.05	h=0.025
$u_1(x)$	MBLM2	0	0	0
	MBLM3	1.1293×10^{-02}	3.7858×10^{-04}	1.9642×10^{-05}
	MBLM5	7.8419×10^{-02}	1.7864×10^{-03}	6.7330×10^{-05}
$u_2(x)$	MBLM2	0	0	0
	MBLM3	1.1167×10^{-02}	1.8703×10^{-04}	6.8744×10^{-06}
	MBLM5	7.8582×10^{-02}	9.7120×10^{-04}	3.7381×10^{-05}

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