The modified decomposition method for analytic treatment of non-linear system of Volterra-Hammerstein Integral Equations

Chinar S. Ahmed[1]
[1]University of Garmian/Kalar – Faculty of Education– Department of Mathematics – Kurdistan Region - Iraq

Abstract.
In this article, non-linear of system Volterra-Hammerstein integral equations of the second kind (*NSVHIEK2*) considered. The modified Adomian decomposition method is used to solve the *NSVHIEK2*. Some illustrative examples are prepared to show the efficiency and simplicity of the method.

Keywords: Modified decomposition method, non-linear system of Volterra-Hammerstein integral equation.

1. Introduction:

Recently a great deal of interest has been focused on the applications of the Adomian method to solve a wide variety of stochastic and deterministic problems (Babolian and Vahidi, 2004). The solution is the sum of an infinite series which converges rapidly to the accurate solutions.

In this paper, we extend the modified decomposition method to solve *NSVHIEK2*. A non-linear system of Volterra-Hammerstein integral equation (Bownds and Wood, 1976) can be written as the following:

$$U(x) = F(x) + \int_0^x k_i(x,t)\left[R(U(x,t)) + N(U(x,t))\right]dt, \qquad (1)$$

where

$$U(x) = (u_1(x), ..., u_m(x))^T,$$
$$F(x) = (f_1(x), ..., f_m(x))^T,$$

and $u_i(x,t)$, $i = 1,2,...,m$ is an unknown function that will be determined, $k_i(x,t)$ is the kernel of the integral equation, $f_i(x,t)$, $i = 1,2,...,m$ is an analytic function, $R(U(x, t))$ and $N(U(x, t))$ are the linear and non-linear system functions of U, respectively.

The concepts of integral equation have attracted much interest in recent years for analytical and numerical treatments. Due to the increased interest in the system of non-linear integral equations, abroad classes of analytical and numerical solution methods have been used to handle these problems (WazWaz, 2005). However, this analytical solution method is not easy to use and require tedious work and knowledge.

To overcome the tedious work involved in the existing strategies and to minimize the proliferation of terms in the Adomian scheme, the modified Adomian decomposition

Corresponding Author: Chinar Sh. Ahmed, University of Garmian/Kalar – Faculty of Education– Department of Mathematics – Kurdistan Region - Iraq
Email: chinarshahab2009@yahoo.com

method is studying.

In this paper, we aim to obtain analytical solutions for *NSVHIEK2*. Although the modified forms introduce a slight change in the formulation of the Adomian recursive relation, it provides a qualitative improvement over standard Adomian method. In addition, the modified technique may give the exact solution for system of non-linear equations without any need for the Adomian polynomials. Although this slight variation is quite simple, it demonstrates the reliability and the power of the modification.

2. The Modified decomposition method:

The standard Adomian method defines the solution $u(x, t)$ by the series

$$u_i(x,t) = \sum_{j=0}^{\infty} u_{ij}(x,t)\,,\ i = 1,2,...,m\,. \tag{2}$$

Substituting the series decomposition (2) into both sides of equation (1) and assuming that the function f_i, $i = 1,2,...,m$ can be expressed as the sum of two parts, f_{i0} and f_{i1}. Therefore we set

$$f_i = f_{i0} + f_{i1}\,,\ i = 1,2,...,m\,. \tag{3}$$

In view of this assumption, we propose a slight variation is that only the part f_{i0} is assigned to the zeroth component u_{i0} and the remaining part f_{i1} is combined with the other terms (integral part) to define u_{i1}, based on the suggestion, we formulate the following modified decomposition method:

$$\sum_{i=0}^{\infty} u_{ii}(x,t) = f_{i0}(x,t) + f_{i1}(x,t) + \int_0^x k_i(x,t)\left[R(\sum_{j=0}^{\infty} u_{ij}(x,t),...,\sum_{j=0}^{\infty} u_{mj}(x,t)) + N(\sum_{i=0}^{\infty} u_{ij}(x,t),...,\sum_{j=0}^{\infty} u_{mj}(x,t)) \right] dt\,. \tag{4}$$

The modified decomposition method introduces the use of the recurrence relation

$$u_{i0}(x,t) = f_{i0}(x,t)\,,$$

$$u_{i1}(x,t) = f_{i1}(x,t) + \int_0^x k_i(x,t)\left[R(u_{10}(x,t),...,u_{m0}(x,t)) + N(u_{10}(x,t),...,u_{m0}(x,t)) \right] dt\,,$$

$$u_{i,(k+2)}(x,t) = \int_0^x k_i(x,t)\left[R(u_{1,(k+1)}(x,t),...,u_{m,(k+1)}(x,t)) + N(u_{1,(k+1)}(x,t),...,u_{m,(k+1)}(x,t)) \right] dt\,,$$

$$k \ge 0\,. \tag{5}$$

As stated before, we may need only two iterations to derive the exact solution. If more than two iteration are needed, $N(U(x, t))$ should be represented by Adomian polynomials which can easily be generated for all types of non-linear systems. The choice of $f_{i0}(x,t)$ contain the minimal number of terms has a strong influence on facilitating the recurrence relation, and as a consequence, accelerates the convergence of the solution. Also, the exact solution must be a term of $f_i(x,t)$ or to be a part of series of $f_i(x,t)$. This means that the success of this method depends mainly on the proper choice of f_{i0} and f_{i1}. We have been unable to establish any criteria to judge what forms of f_{i0} and f_{i1} can be used to yield the acceleration demanded. At present f_{i0} and f_{i1} are selected by trials. Several illustrative examples are used to show the pertinent features of the modified method.

3. The Noise Terms phenomenon:

The noise terms phenomenon gives a useful tool in that, if it appears, it gives a fast convergence of the solution by using two iterations only. It is significant to note that the noise terms may appear only for inhomogeneous problems. It is important to note that these terms may appear for inhomogeneous problems, whereas homogeneous problems do not general noise terms (WazWaz, 2003).

A necessary condition for obtaining exact solution in minimum iteration (only two iteration) is that the exact solution must be a part of noise terms (or a part of series of noise terms). To give a clear overview of the content of this work, several illustrative examples of *NSVHIEK2* have been selected to demonstrate the efficiency of the method and to confirm the necessary condition for obtaining exact solution.

4. Illustrative Examples:

The method of this paper is useful for finding the solutions of NS*VHIEK*2 in terms of modified decomposition method. We illustrate this by solving the following three examples (AL-Asdi, 2002):

Example 1:

Consider a system of non-linear *VHIEK*2's:

$$u_1(x) = 1 + x + \frac{1}{2}\sin(x) + \frac{1}{2}\cos(x) - \frac{1}{2}\exp(x) + \int_0^x (\exp(x-t)\ln\left(u_1(t) + u_2(t)\right))dt$$

$$u_2(x) = (1-x)\sin(x) - \cos(x) - \exp(\sin(x)) + 2 + \int_0^x \cos(t)\ln(u_1(t)) + \exp(u_2(t))\,dt\,.$$

We decompose $f_i(x)$, i=1, 2 as follows:

$f_{10} = \exp(x)$, $f_{20} = \sin(x)$

$f_{11} = 1 + x + \frac{1}{2}\sin(x) + \frac{1}{2}\cos(x) - \frac{3}{2}\exp(x)$, $f_{21} = -x\sin(x) + \cos(x) - \exp(\sin(x)) + 2$.

The modified decomposition technique admits the use of the recursive relation given in the form

$u_{10} = \exp(x)$, $u_{20} = \sin(x)$,

$$u_{11} = x + 1 + \frac{1}{2}\sin(x) + \frac{1}{2}\cos(x) + \frac{1}{2}\exp(x) + \int_0^x \exp(x-t)\left[\ln\left(u_{10}(t)\right) + u_{20}(t)\right]dt\,,$$

$$u_{21} = 2 - x\sin(x) + \cos(x) - \exp(\sin(x)) + \int_0^x \cos(t)\left[\ln\left(u_{10}(t)\right) + \exp(u_{20}(t))\right]dt\,.$$

This gives

$u_{10} = \exp(x)$, $u_{20} = \sin(x)$

$$u_{11} = x + 1 + \frac{1}{2}\sin(x) + \frac{1}{2}\cos(x) + \frac{1}{2}\exp(x) + \int_0^x \exp(x-t)\left[\ln\left(\exp(t)\right) + \sin(t)\right]dt = 0,$$

$$u_{21} = 2 - x\sin(x) + \cos(x) - \exp(\sin(x)) + \int_0^x \cos(t)\left[\ln\left(\exp(t)\right) + \exp(\sin(t))\right]dt = 0.$$

It follows immediately that $u_k = 0,\ k \geq 2$, and the exact solution

$u_1 = \exp(x)$, $u_2 = \sin(x)$

is readily obtained.

Example 2:

Solve a system of non-linear *VHIEK*2's:

$$u_1(x) = 1 + x - \exp(x + \sin(x)) + \int_0^x (\cos(t) + 1)\ \exp\left(u_1(t) + u_2(t)\right)\, dt$$

$$u_2(x) = 1 + \sin(x) - \exp(x\sin(x)) + \int_0^x (t\cos(t) + \sin(t))\exp(u_1(t)u_2(t))\, dt\ .$$

We decompose $f_i(x)$, *i*=1, 2 as follows:

$f_{10} = x$, $f_{20} = \sin(x)$,

$f_{11}(x) = 1 - \exp(x + \sin(x))$, $f_{21}(x) = 1 - \exp(x\sin(x))$.

The modified decomposition technique admits the use of the recursive relation given in the form

$u_{10} = x$, $u_{20} = \sin(x)$,

$$u_{11}(x) = 1 - \exp(x + \sin(x)) + \int_0^x (\cos(t) + 1)\ \exp\left(u_{10}(t) + u_{20}(t)\right)\, dt$$

$$u_{21}(x) = 1 - \exp(x\sin(x)) + \int_0^x (t\cos(t) + \sin(t))\exp(u_{10}(t)u_{20}(t))\, dt\ .$$

This gives

$u_{10} = x$, $u_{20} = \sin(x)$,

$$u_{11}(x) = 1 + x - \exp(x + \sin(x)) + \int_0^x (\cos(t) + 1)\ \exp\left(t + \sin(t)\right)\, dt = 0,$$

$$u_{21}(x) = 1 + \sin(x) - \exp(x\sin(x)) + \int_0^x (t\cos(t) + \sin(t)) + \exp(t\sin(t))\, dt = 0.$$

It follows immediately that $u_{ik} = 0,\ k \geq 2$, *i*=1, 2 and the exact solution

$u_1 = x$, $u_2 = \sin(x)$,

is readily obtained.

Example 3:

Solve a system of non-linear *VHIEK*2's:

$$u_1(x) = \frac{1}{4}(1-\exp(2x)) + \int_0^x (x-t)u_2^2(t)\ dt$$

$$u_2(x) = -x\exp(x) - \exp(x) - 1 + \int_0^x t\exp(-2u_1(t))\,dt\,.$$

We decompose $f_i(x)$, i=1, 2 as follows:

$$f_{10} = -\frac{1}{2}x\,,\ \ f_{20} = \exp(x)\,,$$

$$f_{11} = -\frac{1}{4}\sum_{i=2}^{\infty}\frac{(2x)^i}{i!}\,,\ \ f_{21} = -x\exp x + \exp(x) - 1\,.$$

The modified decomposition technique admits the use of the recursive relation given in the form

$$u_{10} = -\frac{1}{2}x\,,\ u_{20} = \exp(x)\,,$$

$$u_{11}(x) = \frac{1}{4}(1-\exp(2x)) + \frac{1}{2}x + \int_0^x (x-t)u_{20}^2(t)\ dt\,,$$

$$u_{21}(x) = -x\exp(x) + \exp(x) - 1 + \int_0^x t\exp(-2u_{10}(t))\,dt\,.$$

This gives

$$u_{10} = -\frac{1}{2}x\,,\ u_{20} = \exp(x)\,,$$

$$u_{11}(x) = \frac{1}{4}(1-\exp(2x)) + \frac{1}{2}x + \int_0^x (x-t)\exp(2t)\ dt = 0\,,$$

$$u_{21}(x) = -x\exp(x) + \exp(x) - 1 + \int_0^x t\exp(t)\,dt = 0.$$

It follows immediately that $u_{ik} = 0,\ k \geq 2$, i=1, 2 and the exact solution

$$u_1 = -\frac{1}{2}x\,,\ u_2 = \exp(x)\,,$$

is readily obtained.

5. Conclusion:

This paper presents the use of the reliable modified Adomian decomposition method for solving non-linear system Volterra-Hammerstein integral equations of the second kind. The modified Adomian decomposion method is implemented in a straight forward manner and provided significant improvement by requiring only two iterations to obtain the exact solution in most cases. Accelerating the convergence of the modified Adomian method requires that the exact solution must be a part of $f_i(x,t)$ or series of $f_i(x,t)$.

References:

1. **AL-Asdi, A. S.**, The Numerical Solution of Volterra-Hammerstein Second Kind-Integral Equations, M.Sc. thesis, University of AL-Mustansiriya, Iraq, (2002).
2. **Babolian, E.**, **Biazar, J.** and **Vahidi, A. R.,** The Decomposition Method applied to System of Fredholm Integral Equations of the Second Kind, Applied Mathematics and Computation, No. 148 (2004), pp. 443-452.
3. **Bownds, J. M.** and **Wood, B.,** On Numerically Solving Nonlinear Volterra Integral Equation with Fewer Computations, SIAM Journal on Numerical Analysis, Vol. 13, No. 5 (1976), pp. 705-719.
4. **WazWaz, A.-M.**, The modified decomposition method for analytic treatment of non-linear system integral equations and system of non-linear system integral equations**,** International Journal of Computer Mathematics, Vol. 82, No. 9 (2005), pp. 1107-1115.
5. **WazWaz, A.-M.,** The Existence of Noise Terms for Systems of Inhomogeneous Differential and Integral Equations, Applied Mathematics and Computation, No. 146 (2003), pp. 81-92.

معالجة التحليلية لمنظومة معادلات هامرستن فولتيرا التكاملية اللاخطية من النوع الثاني بطريقة تحول التحليلي

چنار شهاب احمد
جامعة صلاح الدين/أربيل – كلية العلوم – قسم الرياضيات

الخلاصة:
في هذه الدراسة، **اعتبر مظومة معا دلات** هامرستن فولتيرا التكاملية اللاخطية من النوع الثاني. طريقة ادوميان التحليلية المتطورة استخدم لحل هذه المعادلات. بعض امثلة ايضاحية اعد لبيان كفاءة و سهولة الطريقة.